# Counting distributions in Bayesian networks and Markov models with polynomials

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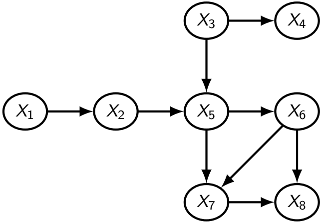
# Bayesian networks - DAG - and belief propagation




$X_{\mathcal{U}} = \{X_u\}_{u \in \mathcal{U}}$
$\forall u \in \mathcal{U}, X_u \in \mathcal{X}_u$

$\mathrm{pa}_u$ : set of labels for parents of $X_u$
$\sum_{X_{\mathcal{U}}} \prod_{u \in \mathcal{U}} \mathbb{P}(X_u | X_{\mathrm{pa}_u}) = 1$

Let $\overline{X_u} = \{X_u, X_{\mathrm{pa}_u}\}$,

$$\mathrm{ev} = \{X_u \in \mathcal{X}_u^* \subset \mathcal{X}_u\} \quad \rightarrow \quad \varphi_{X_u}(\overline{X_u}) = \mathbb{1}_{X_u \in \mathcal{X}_u^*} \mathbb{P}(X_u | X_{\mathrm{pa}_u})$$

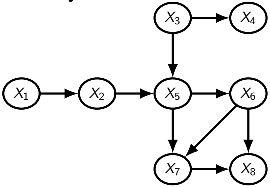
$$\mathbb{P}(\mathrm{ev}) = \sum_{X_{\mathcal{U}}} \prod_{u \in \mathcal{U}} \varphi_{X_u}(\overline{X_u})$$

$$\forall v \in \mathcal{U}, \quad \mathbb{P}(X_v, \mathrm{ev}) = \sum_{X_{\mathcal{U}} \backslash X_v} \prod_{u \in \mathcal{U}} \varphi_{X_u}(\overline{X_u})$$

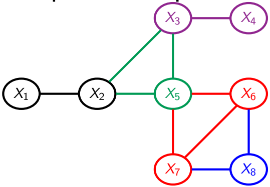
Complexity $= \mathcal{O}\left(\prod_u |\mathcal{X}_u|\right)$, *ie* $\mathcal{O}(2^n)$ for only binary variables.

# Message passing or sum-product algorithm

Bayesian network



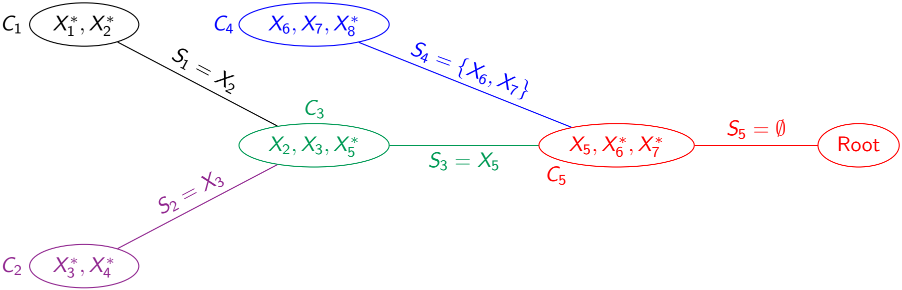

Clique decomposition




$$\mathbb{P}(\mathrm{ev}) = \sum_{X_5}\sum_{X_6}\sum_{X_7}\left\{\sum_{X_8}\varphi_{X_8}(\overline{X_8})\right.$$

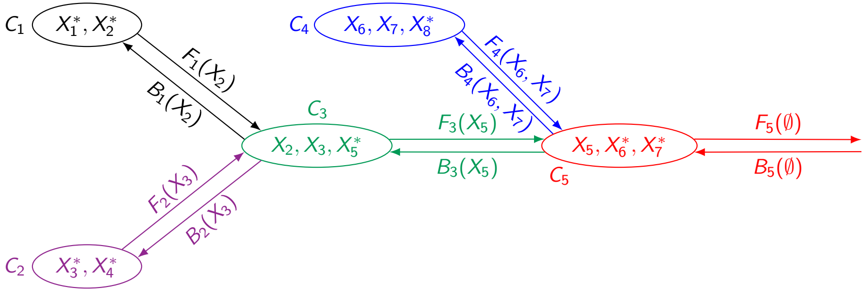
$$\sum_{X_2}\sum_{X_3}\left(\overbrace{\sum_{X_1}\varphi_{X_1}(\overline{X_1})\varphi_{X_2}(\overline{X_2})}^{F_1(X_2)}\overbrace{\sum_{X_4}\varphi_{X_3}(\overline{X_3})\varphi_{X_4}(\overline{X_4})}^{F_2(X_3)}\varphi_{X_5}(\overline{X_5})\right)$$

$$\left.\varphi_{X_6}(\overline{X_6})\varphi_{X_7}(\overline{X_7})\right\}$$




- Junction-tree properties :
  - Tree
  - Running intersection
  - Covering

# Message passing



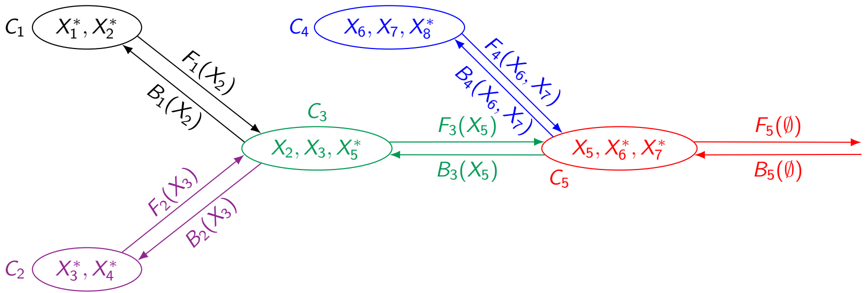

$\forall i \in \{1, \dots K\}; \quad \phi_i(C_i) = \prod_{X_u \in C_i{}^*} \varphi_{X_u}(\overline{X_u})$

$F_i(S_i) = \sum_{C_i \setminus S_i} \left( \prod_{j \in \mathrm{from}_i} F_j(S_j) \right) \times \phi_i(C_i)$

$B_K(\emptyset) = 1$ and for $k \in K, \dots, 1$, $\forall i \in \mathrm{from}_k$,

$B_i(S_i) = \sum_{C_k \setminus S_i} \phi_k(S_k) \times B_k(S_k) \times \prod_{j \in \mathrm{from}_k, j \neq i} F_j(S_j)$

# Message passing



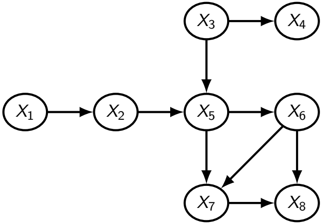

$\forall i \in \{1, \dots, K\}$,

$$\mathbb{P}(S_i, \mathrm{ev}) = F_i(S_i) \times B_i(S_i)$$

$$\mathbb{P}(C_i, \mathrm{ev}) = \phi_i(C_i) \times B_i(S_i) \times \prod_{j \in \mathrm{from}_i, j \neq i} F_j(S_j)$$

Complexity $= \mathcal{O}\left(n \times \text{argmax}_u |\mathcal{X}_u|^{\mathrm{TW}}\right)$, $\mathrm{TW}$ = size of the largest clique

# Idea : Change definition of $\mathrm{ev}$ and work on the arithmetic of potentials to compute other quantities of interest.

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$\forall u \in \mathcal{U}, X_u \in \mathcal{X}_u$

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$\sum_{X_{\mathcal{U}}} \prod_{u\in\mathcal{U}} \mathbb{P}(X_u|X_{\mathrm{pa}_u}) = 1$

Let $\overline{X_u} = \{X_u, X_{\mathrm{pa}_u}\}$

$$\mathrm{ev} = \{X_u \in \mathcal{X}_u^* \subset \mathcal{X}_u\} \quad \rightarrow \quad \varphi_{X_u}(\overline{X_u}) = \mathbb{1}_{X_u\in\mathcal{X}_u^*}\mathbb{P}(X_u|X_{\mathrm{pa}_u})$$

$$\mathbb{P}(\mathrm{ev}) = \sum_{X_{\mathcal{U}}} \prod_{u\in\mathcal{U}} \varphi_{X_u}(\overline{X_u})$$

$$\forall v \in \mathcal{U}, \quad \mathbb{P}(X_v, \mathrm{ev}) = \sum_{X_{\mathcal{U}}\setminus X_v} \prod_{u\in\mathcal{U}} \varphi_{X_u}(\overline{X_u})$$

# Derivatives of the likelihood with derivative generating functions in parametric bayesian networks.

$$\varphi_u(\overline{X_u}) = \mathbb{1}_{X_u \in \mathcal{X}_u^*} \mathbb{P}(X_u | X_{\mathrm{pa}_u}, \theta) \quad \Rightarrow \quad \sum_{X_{\mathcal{U}}} \prod_{u \in \mathcal{U}} \varphi_u(\overline{X_u}) = \mathbb{P}(\mathrm{ev}|\theta) = L(\theta)$$

**Message passing algo.:** $\forall i \in 1, \ldots, K, \quad \sum_{S_i} F_i(S_i) B_i(S_i) = L(\theta)$

with $K$: Number of cliques of the built JT.

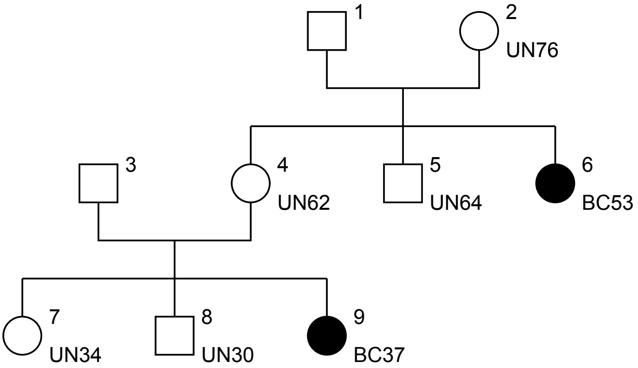
**Polynomial potentials:** $\Phi_u(\overline{X_u}) = \sum_{k=0}^{d} {\varphi_u}^{(d)}(\overline{X_u}) z^d$

**Leibniz's product:** $\sum_{k=0}^{d} a_k z^k \star \sum_{k=0}^{d} b_k z^k = \sum_{k=0}^{d} \sum_{i=0}^{k} \binom{i}{k} a_i b_{k-i} \, z^k$

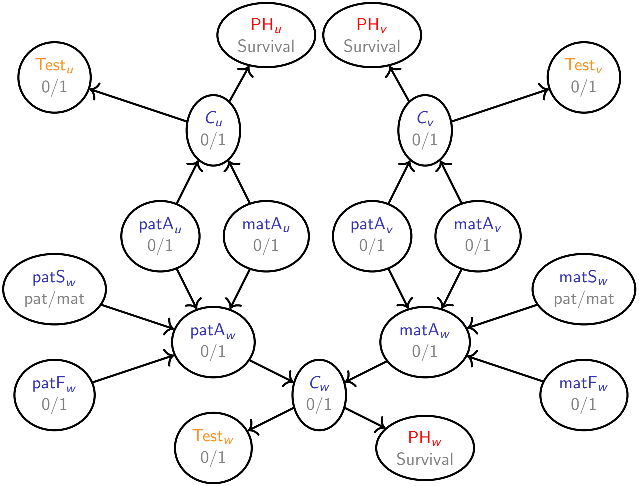
**Message passing algo over polynomial potentials:**

$$\forall i \in 1, \ldots, K, \quad \sum_{S_i} \mathbf{F}_i(S_i) \mathbf{B}_i(S_i) = \sum_{k=0}^{d} L^{(k)}(\theta) \, z^k$$

# Counting distribution with probability and moment generating functions (Application with genetic diseases).




- Distribution of the # of carriers ($N^C$)
- Distribution of the # of filiation errors ($N^F$) (+joint distributions)
- Moments




$$\mathrm{ev} = \{\text{PH}_u\}_{u=1,\dots,n_{\text{ind}}} \cap \{\text{Test}_u\}_{u\in\{1,\dots,n_{\text{ind}}\}}, \quad n_{\text{ind}} = \#\text{ of ind.}$$

# Probability generating function of $N$ : $G_N(z) = \mathbb{E}(z^N)$ with $N = \sum_u C_u$ (Number of carriers).

**Evidence:** $\mathrm{ev} = \{\mathsf{PH}_u\}_{u=1,\dots,n_{\mathrm{ind}}} \cap \{\mathsf{Test}_u\}_{u\in\{1,\dots,n_{\mathrm{ind}}\}}$

**Polynomial Potentials:** $\varphi_{C_u}(\overline{C_u}) = \mathbb{P}(C_u|\mathrm{patA}_u, \mathrm{matA}_u)$

$$\Rightarrow \Phi_{C_u}(\overline{C_u}) = \mathbb{P}(C_u|\mathrm{patA}_u, \mathrm{matA}_u) z^{C_u}$$

For each other variable, $\varphi_{.}(\bar{.}) = \Phi_{.}(\bar{.})$

**Computation with the Sum-product / message passing algorithm:**
With variables renamed $X_u$, $u \in \mathcal{U} = \{1, \dots, n\}$ ($n = \#$ of variables):
$\forall i \in 1, \dots K$ ($K$: # of cliques of the built JT),

$$\sum_s \underbrace{\mathbf{F}_i(s) \times \mathbf{B}_i(s)}_{\text{Polynomial messages}} = \sum_{X_{\mathcal{U}}} \prod_u \Phi_{X_u}(\overline{X_u})$$

$$= \sum_{k\geq 0} \mathbb{P}(N = k, \mathrm{ev}) z^k$$

# multivariate polynomials for joint distributions $N^C = \sum_u C_u$ and $N^F = \sum_u \text{patF}_u$

**Joint distribution of # of carriers and # of paternal filiation errors**

For all individuals: $\varphi_{C_u}(\overline{C_u}) \Rightarrow \Phi_{C_u}(\overline{C_u}) = \mathbb{P}(C_u|\text{patA}_u, \text{matA}_u) z^{C_u}$

For offsprings: $\varphi_{\text{patF}_u}(\overline{\text{patF}_u}) \Rightarrow \Phi_{\text{patF}_u}(\overline{\text{patF}_u}) = \mathbb{P}(\text{patF}_u) y^{\text{patF}_u}$

**Computation with the sum-product / message-passing algorithm:**
With variables renamed $X_u$, $u \in \mathcal{U} = \{1, \ldots, n\}$ ($n = \#$ of variables):
$\forall i \in 1, \ldots K$ ($K$: # of cliques of the built JT),

$$\sum_s \underbrace{\mathbf{F}_i(s) \times \mathbf{B}_i(s)}_{\text{Polynomial messages}} = \sum_{X_\mathcal{U}} \prod_u \Phi_{X_u}(\overline{X_u})$$

$$= \sum_{k \geq 0} \sum_{\ell \geq 0} \mathbb{P}(N^C = k, N^F = \ell, \text{ev}) z^k y^\ell$$

# Moment generating function of $N$ : $M_N(t) = \mathbb{E}(e^{tN})$

**Polynomial potentials:** $\varphi_{C_u}(\overline{C_u}) = \mathbb{P}(C_u|\text{patA}_u, \text{matA}_u)$

$$\Rightarrow \Phi_{C_u}(\overline{C_u}) = \mathbb{P}(C_u|\text{patA}_u, \text{matA}_u)e^{tC_u}$$

$$\Rightarrow \Phi_{C_u}(\overline{C_u}) = \mathbb{P}(C_u|\text{patA}_u, \text{matA}_u)\left[\sum_{\ell=0}^{d}\frac{t^\ell}{\ell!}\right]^{C_u}$$ with $d$, a chosen order.

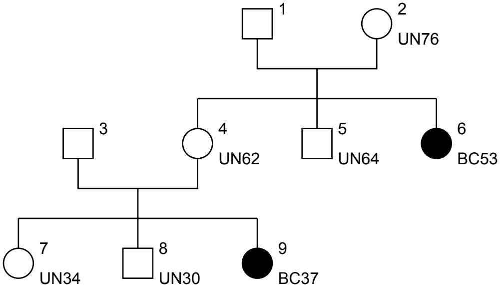
For each other variable, $\varphi_{.}(\bar{.}) = \Phi_{.}(\bar{.})$

**Computation with the sum-product / message-passing algorithm:**
$\forall i \in 1, \ldots K,$

$$\sum_s \underbrace{\mathbf{F}_i(s) \times \mathbf{B}_i(s)}_{\textbf{Polynomial messages}} = \sum_{m \geq 0} \mathbb{P}(N = m, \text{ev})e^{mt}$$

$$= \sum_{m \geq 0}\sum_{\ell=0}^{d} \mathbb{P}(N = m, \text{ev})\frac{m^\ell t^\ell}{\ell!} = \sum_{\ell=0}^{d} \underbrace{\left\{\sum_{m \geq 0} \mathbb{P}(N = m, \text{ev})m^\ell\right\}}_{\mathbb{E}[N^\ell, \text{ev}]}\frac{t^\ell}{\ell!}$$

# Results



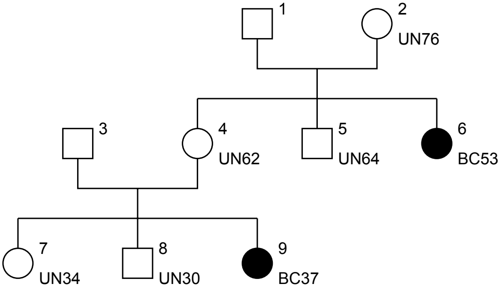

$$\pi(\cdot) = \mathbb{P}(\cdot|\mathrm{FH})$$

| 2/3 | 2/4 | 2/6 | 2/9 | $\pi(C_{\mathcal{J}})$ |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 67.5 |
| 0 | 1 | 1 | 1 | 18.0 |
| 1 | 0 | 0 | 1 | 10.0 |
| 0 | 0 | 1 | 0 | 1.9 |
| 0 | 1 | 0 | 1 | 1.4 |
| 0 | 1 | 1 | 0 | 0.4 |

| Individual *i* | $\pi(\mathrm{N})$ | 2/1 | 2/2 | 2/3 | 2/4 | 2/5 | 2/6 | 2/7 | 2/8 | 2/9 |
|---|---|---|---|---|---|---|---|---|---|---|
| $\pi(C_i = 1)$ | – | 17.3 | 5.1 | 10.7 | 20.0 | 11.1 | 20.8 | 14.7 | 15.2 | 30.0 |
| $\pi(C_i = 1 \mid \mathrm{N} = 0)$ | 67.3 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| $\pi(C_i = 1 \mid \mathrm{N} = 5)$ | 7.6 | **77.6** | 22.4 | 1.6 | **98.5** | 36.9 | **93.1** | 34.7 | 37.0 | **98.1** |
| $\pi(C_i = 1 \mid \mathrm{N} = 6)$ | 7.0 | **77.6** | 22.6 | 2.1 | **98.5** | 68.1 | **97.5** | 66.0 | 68.3 | **99.3** |
| $\pi(C_i = 1 \mid \mathrm{N} = 3)$ | 6.2 | 14.6 | 4.2 | **81.1** | 4.1 | 14.8 | 15.6 | 39.7 | 42.5 | **83.5** |
| $\pi(C_i = 1 \mid \mathrm{N} = 4)$ | 5.5 | 44.0 | 12.6 | 44.2 | 55.7 | 4.4 | 46.6 | 47.5 | 47.8 | **97.1** |
| $\pi(C_i = 1 \mid \mathrm{N} = 2)$ | 3.8 | 20.9 | 5.9 | 73.1 | 0.3 | 1.9 | 24.8 | 1.4 | 1.5 | 70.4 |

3 founders, 6 offsprings, 60 variables, 41 cliques, complexity 748

# Results



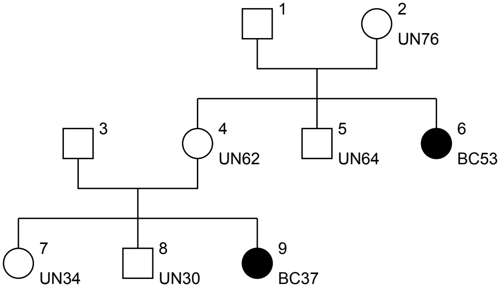

$\pi(\cdot) = \mathbb{P}(\cdot|\mathrm{FH}, \mathrm{T}_4 = 1)$

| 2/3 | 2/4 | 2/6 | 2/9 | $\pi(\mathrm{C}_{\mathcal{J}})$ |
|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 81.8 |
| 0 | 0 | 0 | 0 | 7.7 |
| 0 | 1 | 0 | 1 | 6.4 |
| 0 | 1 | 1 | 0 | 1.7 |
| 1 | 0 | 0 | 1 | 1.1 |
| 1 | 1 | 1 | 1 | 0.8 |

| Individual $i$ | $\pi(\mathrm{N})$ | 2/1 | 2/2 | 2/3 | 2/4 | 2/5 | 2/6 | 2/7 | 2/8 | 2/9 |
|---|---|---|---|---|---|---|---|---|---|---|
| $\pi(\mathrm{C}_i = 1)$ | – | 70.8 | 20.7 | 2.0 | **90.9** | 45.4 | **84.6** | 44.6 | 46.2 | **90.2** |
| $\pi(\mathrm{C}_i = 1\|\mathrm{N} = 5)$ | 34.1 | **77.6** | 22.4 | 0.2 | **100** | 36.9 | **93.1** | 34.7 | 37.0 | **98.1** |
| $\pi(\mathrm{C}_i = 1\|\mathrm{N} = 6)$ | 31.3 | **77.6** | 22.6 | 0.7 | **100** | 68.0 | **97.5** | 66.0 | 68.3 | **99.3** |
| $\pi(\mathrm{C}_i = 1\|\mathrm{N} = 4)$ | 14.1 | **76.2** | 21.8 | 2.0 | **98.1** | 7.6 | **80.5** | 9.2 | 9.7 | **95.0** |
| $\pi(\mathrm{C}_i = 1\|\mathrm{N} = 7)$ | 10.2 | **77.7** | 23.6 | 3.8 | **100** | **97.4** | **99.8** | **98.9** | **99.0** | **100** |
| $\pi(\mathrm{C}_i = 1\|\mathrm{N} = 0)$ | 7.6 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| $\pi(\mathrm{C}_i = 1\|\mathrm{N} = 3)$ | 1.8 | 51.5 | 14.6 | 31.2 | 63.1 | 6.6 | 18.0 | 17.4 | 18.7 | **78.9** |

3 founders, 6 offsprings, 60 variables, 41 cliques, complexity 748

# Results




$$\pi(\cdot) = \mathbb{P}(\cdot | \mathrm{FH}, \mathsf{T}_4 = 0)$$

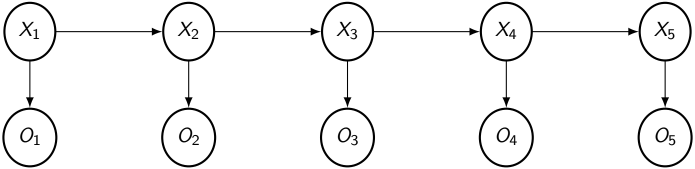
| 2/3 | 2/4 | 2/6 | 2/9 | $\pi(C_{\mathcal{J}})$ |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 80.3 |
| 1 | 0 | 0 | 1 | 11.9 |
| 0 | 1 | 1 | 1 | 4.4 |
| 0 | 0 | 1 | 0 | 2.2 |
| 0 | 1 | 0 | 1 | 0.3 |
| 1 | 0 | 1 | 1 | 0.3 |

| Individual $i$ | $\pi(\mathrm{N})$ | 2/1 | 2/2 | 2/3 | 2/4 | 2/5 | 2/6 | 2/7 | 2/8 | 2/9 |
|---|---|---|---|---|---|---|---|---|---|---|
| $\pi(C_i = 1)$ | – | 5.9 | 1.7 | 12.5 | 4.9 | 3.8 | 7.1 | 8.3 | 8.6 | 17.1 |
| $\pi(C_i = 1 \| \mathrm{N} = 0)$ | 80.1 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| $\pi(C_i = 1 \| \mathrm{N} = 3)$ | 7.1 | 12.6 | 3.6 | **83.9** | 0.9 | 15.2 | 15.5 | 40.9 | 43.8 | **83.7** |
| $\pi(C_i = 1 \| \mathrm{N} = 2)$ | 4.5 | 20.9 | 5.9 | 73.2 | 0.1 | 1.9 | 24.9 | 1.4 | 1.5 | 70.4 |
| $\pi(C_i = 1 \| \mathrm{N} = 4)$ | 3.6 | 17.2 | 4.9 | **79.5** | 20.4 | 1.8 | 18.2 | **79.5** | **79.6** | **98.8** |
| $\pi(C_i = 1 \| \mathrm{N} = 5)$ | 2.0 | **77.6** | 22.4 | 7.2 | **93.0** | 36.9 | **93.1** | 34.8 | 37.1 | **98.1** |
| $\pi(C_i = 1 \| \mathrm{N} = 6)$ | 1.8 | **77.6** | 22.6 | 7.5 | **93.1** | 68.1 | **97.5** | 66.0 | 68.3 | **99.3** |

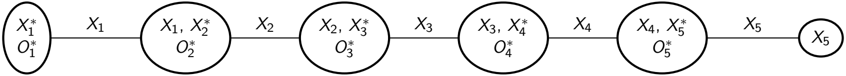
3 founders, 6 offsprings, 60 variables, 41 cliques, complexity 748

# Number of errors in a Markov Chain, $N = \sum_i \mathbb{1}_{X_i \neq O_i}$

## Hidden Markov chain




## Junction tree




## Potentials

$$\varphi_{X_1}(X_1) = \mu_1;\ \varphi_{X_i}(\overline{X_i}) = \pi(X_{i-1}, X_i) \text{ with } \pi = \begin{pmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{pmatrix}$$

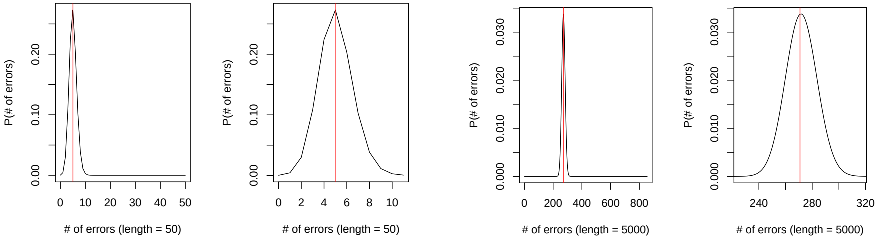
PGF: $\varphi_{O_i}(O_i, X_i = j) = \underbrace{\mathbb{1}_{O_i=\omega_i}}_{\text{ev}} (1-\eta)^{\mathbb{1}_{O_i=j}} (\eta \times z)^{\mathbb{1}_{O_i \neq j}}$ with $\eta = 0.05$

MGF: $\varphi_{O_i}(O_i, X_i = j) = \overbrace{\mathbb{1}_{O_i=\omega_i}} (1-\eta)^{\mathbb{1}_{O_i=j}} \left(\eta(1 + t + t^2/2 + \ldots)\right)^{\mathbb{1}_{O_i \neq j}}$

$$\text{PGF: } \forall i \in 1, \dots K, \quad \sum_{X_i} \mathbf{F}_i(X_i)\mathbf{B}_i(X_i) = \sum_{k \geq 0} \mathbb{P}(N = k, \text{ev}) z^k$$

$$\text{MGF: } \forall i \in 1, \dots K, \quad \sum_{X_i} \mathbf{F}_i(X_i)\mathbf{B}_i(X_i) = \sum_{\ell \geq 0} \mathbb{E}[N^\ell, \text{ev}] \frac{t^\ell}{\ell!}$$

**Distributions:**




**Moments:**

$m_0 = 1.00$, $m_1 = 5.04$, $m_2 = 27.63$    $m_0 = 1.00$, $m_1 = 270.85$, $m_2 = 73501.20$

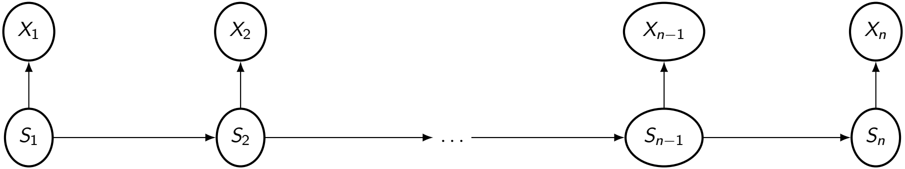
# Number of atypical segments in a sequence
# Joint work with S. Mercier

Observed variables: $X = X_1, \ldots, X_n \in \mathcal{X}^n$ (*e.g.* $\mathcal{X} = \{\text{A, C, G, T}\}$)

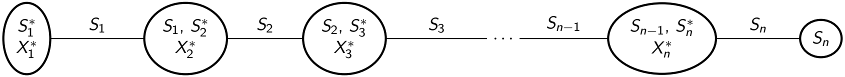
Latent variables: $S = S_1, \ldots, S_n \in \mathcal{S}^n$ ($\mathcal{S} = \{1, 2, 3\}$)

Emission: $\mathbb{P}(X_i|S_i) = \begin{cases} q_0(X_i) \text{ if } S_i \neq 2 \\ q_1(X_i) \text{ if } S_i = 2 \end{cases}$ Constr. transition $\pi = \begin{pmatrix} 1\,1\,0 \\ 0\,1\,1 \\ 0\,0\,1 \end{pmatrix}$

**DAG**




**Junction tree**



- **Evidence:** $\forall x \in \mathcal{X}$, $e(x) = \frac{q_1(x)}{q_0(x)}$ or $e(x) = \exp\left(\frac{f(x)}{T}\right)$
- **Potentials:**

$$\phi_1(S_1 = k) = \pi(1, k)e(x_1)^{\mathbb{1}_{k=2}}$$

$$\text{For } i = 2, \dots, n \quad \phi_i(S_{i-1} = j, S_i = k) = \pi(j, k)e(x_i)^{\mathbb{1}_{k=2}}$$

- **Forward - Backward:**

$$F_1(S_1) = \phi_1(S_1) \quad \text{and for } i = 2, \dots, n \quad F_i(k) = \sum_j F_{i-1}(j)\phi_i(j, k)$$

$$B_n(\emptyset) = 1 \text{ (conv.)} \quad \text{and for } i = n, \dots, 2, \quad B_{i-1}(j) = \sum_k \phi(j, k)B_i(k)$$

- **Start and stop of a segment**

$$\text{P(Start at pos. } i\text{): } \mathbb{P}(S_i = 2, S_{i-1} = 1, X) \propto F_{i-1}(1)\phi_i(1, 2)B_i(2)$$

$$\text{P(Stop at pos. } i\text{): } \mathbb{P}(S_i = 2, S_{i+1} = 3, X) \propto F_i(2)\phi_i(2, 3)B_{i+1}(3)$$

- **Probability of the sequence** $\forall i \in \{1, \dots, n\}$ $\mathbb{P}(X) \propto \sum_k F_i(k)B_i(k)$

# $N$ segments

**First approach**

$\mathcal{S} = \{1, 2, 3, 4, 5, \dots, 2N, 2N+1\}$ and $\pi$ of dimension $(2N+1) \times (2N+1)$

**With polynomials, pgf & mgf**: $\mathcal{S} = \{0, 1\}$ and $\pi = \begin{pmatrix} 1 & z \\ 1 & 1 \end{pmatrix}$

$$\phi_1(S_1) = e(x_1)^{\mathbb{1}_{S_1}} \pi(S_0, S_1) \quad \text{with } S_0 = 0 \text{ (conv.)}$$

$$\forall i = 2, \dots, n \quad \phi_i(j, k) = e(x_i)^{\mathbb{1}_{S_i}} \pi(j, k)$$

**Forward-Backward algo** $\rightarrow$

$$\forall i \in \{1, \dots, n\} \quad F_i(S_i) B_i(S_i) \propto \sum_k \mathbb{Q}(S_i, N = k, X) z^k$$

and $\mathbb{P}(S_i, N = k, X) \propto \mathbb{Q}(S_i, N = k, X) / \binom{n+1}{2k}$

# Start / Stop positions with $N \geq 0$, Distibution of $N$

$$\mathbb{Q}(N = k, X) \propto \Big[\sum_j F_i(j) B_i(j)\Big]_{z^k}$$

and

$$\mathbb{P}(X) \propto \sum_k \mathbb{Q}(N = k, X)$$

$\mathbb{P}$(**any segment starts at position i**)

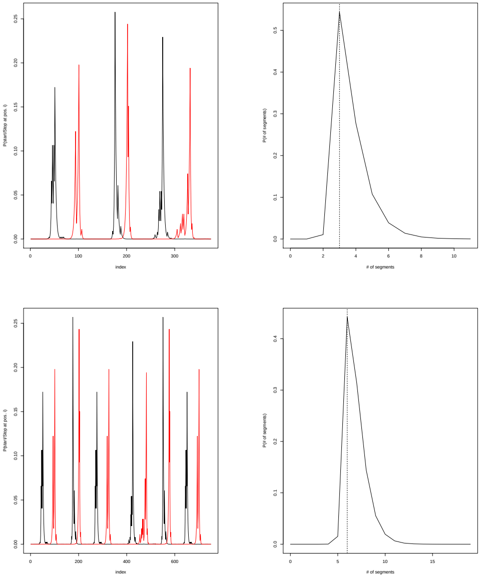
$$\mathbb{P}(S_{i-1} = 0, S_i = 1 | N = k, X) = \Big[F_{i-1}(0) B_i(1) e(X_i) \times z\Big]_{z^k} / \mathbb{Q}(N = k, X)$$

$\mathbb{P}$(**any segment stops at position i**)

$$\mathbb{P}(S_i = 1, S_{i+1} = 0 | N = k, X) = \Big[F_i(1) B_{i+1}(0)\Big]_{z^k} / \mathbb{Q}(N = k, X)$$

**Distribution of $N$**

$$\mathbb{P}(N = k | X) = \frac{\mathbb{Q}(N = k, X)}{\binom{n+1}{2k} \times \mathbb{P}(X)}$$




top : simu. with 3 atypical seg. bottom: simu. with 6 atypical seg.